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How Likely Is Polya's Drunkard to Stay in $x \geqslant y \geqslant z$?

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In his celebrated paper, Polya has considered the random walk in the three-dimensional (cubic) lattice and showed that the probability of return to the origin is less than 1. Subsequent authors have shown that the probability is %34.053.... Here we consider the same random walk, with the restriction that the drunkard is only allowed to stay in $x \geqslant y \geqslant z$. It is shown that his probability of returning to the origin *and staying in the allowed region* is %6.4844....

KEY WORDS: Restricted random walk; method of images; return probability; linear partial recurrence equations.

1. INTRODUCTION AND RESULTS

We will consider the customary (discrete time) random walk in the three-dimensional cubic lattice. Here a particle is allowed to make any one of the six steps $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$ with equal probability. Let a_n be the number of ways of going from the origin back to the origin in $2n$ steps. Polya[1] showed that the probability $a_n/6^{2n}$ that the particle will return to the origin after $2n$ steps is $\leqslant C/n^{3/2}$, for some constant C independent of n, and thus that the expected number of visits to the origin $m = \sum_0^\infty a_n/6^{2n}$ is finite, and thus that the probability of ever returning to the origin, $u = (m-1)/m$, is less than 1. Subsequent authors[2-5] (see also p. 126 in Doyle and Snell's book[6]) have found that $m = 1.516386059137...$, and thus $u = 0.340537329544...$.

The following results concerning a_n are either well known[7,8] or easily derivable from well-known results. Theorem 4 seems to be new.

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Theorem 1:

$$a_n = \sum_{k=0}^{n} \frac{(2n)!\,(2k)!}{(n-k)!^2\,k!^4}$$

Theorem 2:

$$\sum_{n=0}^{\infty} \frac{(-1)^n a_n}{(2n)!} t^{2n} = J_0(-2it)^3$$

Theorem 3.

$$a_n/6^{2n} \sim [\tfrac{1}{4}(3/\pi)^{3/2}]\,n^{-3/2}$$

Theorem 4:

$$36(n+1)(2n+3)(2n+1)a_n - 2(2n+3)(10n^2+30n+23)a_{n+1} + (n+2)^3 a_{n+2} = 0$$

In this paper we will give counterparts to the above theorems for $\bar{a}_n$, the number of ways of returning to origin after $2n$ steps and *always staying within* $x \geqslant y \geqslant z$. In other words, we assume that the "walls" $x-y=-1$ and $y-z=-1$ are "absorbing" and the drunkard dies if he bumps into a wall. Such restricted random walks have been studied by Huse *et al.*[9,10] and by Fisher,[11] in order to model commensurate melting and wetting, and to study dislocation.

The following results will be proved.

Theorem A:

$$\bar{a}_n = \sum_{k=0}^{n} \frac{(2n)!\,(2k)!}{(n-k)!\,(n+1-k)!\,k!^2\,(k+1)!^2}$$

Theorem B:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \bar{a}_n}{(2n)!} t^{2n} = \det[J_{l-k}(-2it)]_{1 \leqslant l,k \leqslant 3}$$

Theorem C:

$$\frac{\bar{a}_n}{6^{2n}} \sim \left(\frac{3^{9/2}}{16\pi^{3/2}}\right) n^{-9/2}$$

Theorem D:

$$
\begin{aligned}
&-72(n+2)(n+1)(2n+9)(2n+5)(2n+3)(2n+1)\bar{a}_n \\
&\quad + 4(n+2)(2n+5)(2n+3)(38n^3+381n^2+1252n+1377)\bar{a}_{n+1} \\
&\quad - 2(n+3)(2n+5)(22n^2+145n+229)(4+n)^2\bar{a}_{n+2} \\
&\quad + (2n+7)(n+3)(n+5)^2(4+n)^2\bar{a}_{n+3} = 0
\end{aligned}
$$

2. PROOFS OF THEOREMS 1–4

For the sake of completeness, we will first sketch the proofs of Theorems 1–4.

Sketch of Proof of Theorem 1. For the walker in three dimensions to return, the walker must take an equal number of steps in the different directions.[6] Thus, we have

$$
a_n = \sum_{j,k} \frac{(2n)!}{j!\, j!\, k!\, k!\, (n-j-k)!\, (n-j-k)!}
$$

The summation over j is evaluated exactly by the Vandermonde–Chu identity,[12] and we get the single sum of theorem 1. ∎

We will need the following lemma.

Lemma 1. Let $p(\alpha, \beta, \gamma; a, b, c; n)$ be the number of possible walks, with n steps, from (α, β, γ) to (a, b, c). For any Laurent polynomial F, let $CT\,F$ stand for coefficient of $x^0y^0z^0$ in F. We have

$$
p(\alpha, \beta, \gamma; a, b, c; n) = CT\,\frac{(x+x^{-1}+y+y^{-1}+z+z^{-1})^n}{x^{a-\alpha}y^{b-\beta}z^{c-\gamma}}
$$

Proof. Without loss of generality, we can take $(\alpha, \beta, \gamma) = (0, 0, 0)$. Expanding out $(x+x^{-1}+y+y^{-1}+z+z^{-1})^n$ simulates all possible walks, and the number of those that wind up at (a, b, c) is exactly the coefficient of $x^ay^bz^c$. ∎

Recall that the Bessel functions $J_a(x)$ have the generating function[13]

$$
\sum_{a=-\infty}^{\infty} J_a(z)\, w^a = \exp[(z/2)(w - w^{-1})] \tag{1}
$$

Proof of Theorem 2. By Lemma 1, we have that

$$
a_n = CT(x+x^{-1}+y+y^{-1}+z+z^{-1})^{2n}
$$

Thus,

$$\sum_{n=0}^{\infty} \frac{(-1)^n a_n}{(2n)!} t^{2n}$$

$$= \sum_{n=0}^{\infty} CT \frac{(-1)^n t^{2n}(x+x^{-1}+y+y^{-1}+z+z^{-1})^{2n}}{(2n)!}$$

$$= CT \sum_{n=0}^{\infty} \frac{(-it)^{2n}(x+x^{-1}+y+y^{-1}+z+z^{-1})^{2n}}{(2n)!}$$

$$= CT \sum_{n=0}^{\infty} \frac{(-it)^{n}(x+x^{-1}+y+y^{-1}+z+{}^{-1})^{n}}{n!}$$

since $CT(x+x^{-1}+y+y^{-1}+z+z^{-1})^{2n+1}=0$. But the right side is equal to

$$CT \exp[(x+x^{-1}+y+y^{-1}+z+z^{-1})(-it)]$$
$$= CT \exp[(x+x^{-1})(-it)] \exp[(y+y^{-1})(-it)] \exp[(z+z^{-1})(-it)]$$

By (1) this is equal to

$$CT\left(\sum_a J_a(-2it)(-ix)^a\right)\left(\sum_b J_b(-2it)(-iy)^b\right)\left(\sum_c J_c(-2it)(-iz)^c\right)$$
$$= J_0(-2it)^3 \quad \blacksquare$$

Replacing CT above by the coefficient of $x^{a-\alpha} y^{b-\beta} z^{c-\gamma}$, we have the following result.

Lemma 2:

$$\sum_{n=0}^{\infty} p(\alpha, \beta, \gamma; a, b, c; n) \frac{(-1)^n t^n}{n!}$$
$$= (-i)^{(a+b+c-\alpha-\beta-\gamma)} J_{a-\alpha}(-2it) J_{b-\beta}(-2it) J_{c-\gamma}(-2it) \quad \blacksquare$$

Sketch of the Proof of Theorem 3. Theorem 3 can be obtained by using a method described in pp. 65–67 of Knuth's book,[14] but it may also be shown directly using a result of Wimp,[15]

$${}_3F_2\left(\begin{matrix}-n, -n-\lambda, \alpha+1\\ \lambda+1, \alpha+\beta+2\end{matrix}; w\right)$$
$$\sim \frac{\Gamma(\alpha+\beta+2)\,\Gamma(\lambda+1)}{\Gamma(\alpha+1)\,2\pi^{1/2}} w^{-\lambda/2-\beta/2-3/4}(1+\sqrt{w})^{2n+2\lambda+\beta+2} n^{-\lambda-\beta-3/2},$$
$$n \to \infty, \quad \alpha, \beta, \lambda > -1, \quad w > 0$$

and noticing that a_n may be written as a hypergeometric series

$$a_n = \frac{4^n(1/2)_n}{n!}\, {}_3F_2\left(\begin{matrix} -n,\ -n,\ 1/2 \\ 1,\ 1 \end{matrix}; 4\right) \quad \blacksquare$$

Sketch of the Proof of Theorem 4. Zeilberger[16] proved that every binomial coefficient sum satisfies a homogeneous linear recurrence equation with polynomial coefficients, and he also developed an efficient algorithm for finding and *proving* it.[17] A MAPLE program implementing this algorithm is available from Zeilberger upon request. Theorem 4 was discovered and *proved* by this program. Since MAPLE is readily available, and the interested reader can request the program, there is no point in reproducing the proof. Furthermore, readers who have MACSYMA can prove Theorem 4 for themselves as follows. Let $F(n, k)$ be the summand on the right side of Theorem 1. Evaluate the expression obtained by replacing a_n on the left of Theorem 4 by $F(n, k)$, and apply Gosper's command *nusum* (with respect to k) to it, and call the result $G(n, k)$. Theorem 4 follows upon observing that $G(n, k)$ vanishes for large $|k|$. $\blacksquare$

3. PROOFS OF THEOREMS A–D

We will start by proving Theorem B. We first need the following result.

Lemma 3. Let $q(a, b, c; n)$ be the number of ways of walking n steps from $(0, 0, 0)$ to (a, b, c), and staying within $x \geqslant y \geqslant z$; then

$$q(a, b, c; n) = CT[(1 - x/y)(1 - x/z)(1 - y/z)(x + x^{-1} + y + y^{-1} + z + z^{-1})^n / x^a y^b z^c] \tag{2}$$

Proof. This can be proved using "Kelvin's method of images," as given by formula (5.9), p. 682, of Fisher's paper.[11] Alternatively, if follows from the observation that both sides of (2) satisfy the recurrence

$$\begin{aligned} F(a, b, c; n) = {} & F(a-1, b, c; n-1) + F(a+1, b, c; n-1) \\ & + F(a, b-1, c; n-1) + F(a, b+1, c; n-1) \\ & + F(a, b, c-1; n-1) + F(a, b, c+1; n-1) \end{aligned}$$

subject to the initial and boundary conditions that uniquely determine it:

$$F(a, b, c; 0) = \delta_{a,0}\, \delta_{b,0}\, \delta_{c,0},$$

$$F(a, b, c; n) \equiv 0 \quad \text{on} \quad a - b = -1 \text{ and } b - c = -1 \quad \blacksquare$$

Proof of Theorem B. Expand $(1-x/y)(1-x/z)(1-y/z)$ inside the expression of Lemma 3 for $\bar{a}_n = q(0, 0, 0; 2n)$, take the exponential generating function, and use Lemma 2. ▮

Proof of Theorem A. Let $D(t)$ be the determinantal expression on the right of Theorem B. Writing $J_{-1} = -J_1$ and $J_{-2} = J_2$ and expressing J_2 in terms of J_0 and J_1 gives[13]

$$(2/it)\, J_1(2it)[J_1^2(2it) + J_0^2(2it) - (1/2it)\, J_0(2it)\, J_1(2it)]$$

Now the product in the brackets may be written as hypergeometric functions ${}_1F_2$.[18] We get, on combining terms,

$$\begin{aligned}&(1/2it)\, J_1^2(2it) - (1/2it)\, J_0^2(2it) + [1/(4t^2)]\, J_0(2it)\, J_1(2it)\\ &\quad = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k (1/2)_k (2it)^{2k-1}}{k!\,(2)_k}\end{aligned}$$

Now, multiplying this by

$$4J_1(2it) = 4it \sum_{k=0}^{\infty} \frac{(-1)^k (it)^{2k}}{k!\,(2)_k}$$

and collecting the coefficient of t^{2n} yields Theorem A.

Sketch of the Proof of Theorem C. Same as the proof of Theorem 3. ▮

Sketch of the Proof of Theorem D. Same as the proof of Theorem 4. ▮

To find $\bar{m}$, the expected number of visits at the origin, we computed $\bar{a}_n$ for $n = 0 \ldots 1000$, using Theorem D, and summed $\bar{a}_n/6^{2n}$. By Theorem C the terms grow like $C/n^{9/2}$, and thus the error committed by quitting after N terms is $C/N^{7/2}$, which for $N = 1000$ gives well over nine digits after the decimal point. We got $\bar{m} = 1.0693411\ldots$.

Let $\bar{u}$ be the probability that the particle will ever return to the starting point, staying within $x \geqslant y \geqslant z$. Then, of course, $\bar{m} = 1 + \bar{u} + \bar{u}^2 + \cdots = 1/(1-\bar{u})$, and thus $\bar{u} = 1 - 1/\bar{m} = 0.06484471\ldots$.

4. CONCLUSION

We have used an interplay of various techniques to give precise quantitative information about the random walk in the region $x \geqslant y \geqslant z$. The techniques we have used are: the method of images, partial difference equations, exponential generating functions, asymptotics of hypergeometric functions, and Zeilberger's program for generating and proving linear recurrences satisfied by binomial coefficient sums. In order to get the asymptotic formula of Theorem C and the recurrence of Theorem D, we needed first the explicit expression of Theorem A. Theorem A, in turn, was proved using the exponential generating function of Theorem B. The present methods should extend to higher dimensions and different regions and lattices.

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